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Topics on Space-Time Topology II

GEORGE S. WHISTON

Department of Mathematics, The University of Durham, South Road, Durham, England

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Abstract

Closed, compact, oriented, Lorentzian four-manifolds are investigated using elementary cobordism theory. The groups of cobordism classes of such manifolds under various cobordism relations are calculated. Oriented vector cobordism classes form an infinite free cyclic group and oriented cobordism classes form a subgroup of index two in the four-dimensional oriented cobordism group. The properties of compact five-manifolds bounded by closed, compact, oriented, Lorentzian four-manifolds are investigated and some speculations made on their possible interpretation.

In Whiston (1974) it was demonstrated that any closed, compact, orientable space-time is cobordant in the unoriented sense, that is, bounds a compact five-manifold. It could be asserted that this result is intuitively obvious: 'if a manifold is compact (finite), closed (unbounded) and orientable, it must "close in on itself" and therefore represent either the inside or the outside edge of "something"'. This intuitive reasoning (based on the properties of two-manifolds) is fallacious as there exist compact, closed, orientable four-manifolds which are not boundaries (the complex projective plane $\mathbb{C}P^2$ is one example). The extra structure imposed on a space-time (the Lorentzian structure) ensures that it is sufficiently simple topologically for it to conform to the above reasoning. Therefore if space-time is closely modeled by a closed, compact, orientable, smooth Lorentzian four-manifold it must be the boundary of some compact five-manifold: a 'Hyperspace'. Hyperspaces are therefore as 'reasonable' as the structures imposed on a space-time, so we examine the reasonability of the latter structures. Firstly, unbounded space-times are aesthetically pleasing since they are topologically homogeneous (boundary points are distinguished in a bounded manifold). The orientability of a space-time is guaranteed if the reasonable conditions of time and space-orientability are fulfilled. In fact, if the CPT theorem has any global validity, time reversals and space inversions must occur simultaneously, which is sufficient for space-time to be an orientable manifold. The smooth Lorentzian structure is a requirement of being able to determine the local inertial frames in a consistent smooth way over the whole space-time. Compactness is probably the least justifiable property. This is because compact, time-orientable space-times have closed trips, violating causality. However, as yet, there is no observational evidence that Space-Time is non-compact or that it is time-orientable. Therefore, compact, closed orientable, smooth, Lorentzian four-manifolds still represent a possible model for Space-Time and the 'wormhole trips through higher dimensions' and the 'stargates' so beloved by the science-fiction writers have at least a mathematical existence. There is no unique choice of a hyperspace for a given space-time, for by plumbing on a closed, compact five-manifold to a hyperspace, one obtains another. By using more complicated variants of ordinary unoriented cobordism theory, one can ask if there exist hyperspaces with certain geometric structures (such as orientations or spinor-structures) which extend similar structures off the boundary space-time. In this article, we propose to examine the possible types of hyperspaces that a space-time can bound using the cobordism theories mentioned above. Cobordism is also a classificational tool and we shall use it to classify compact, closed, orientable space-time up to the cobordisms.

Perhaps the central problem of space-time geometry is to classify all possible Lorentzian structures on smooth four-manifolds. A subproblem is, of course, to determine which (smooth) four-manifolds can admit a Lorentzian structure. The latter problem was solved by Steenrod (1951) and by Markus (1955). Any non-compact four-manifold admits a Lorentzian structure whilst a compact four-manifold can admit one iff it has trivial Euler number. The result was proved in the following way. Recall that a (smooth) Lorentzian structure on a (smooth) four-manifold X is a reduction of the Einstein (frame) bundle $GL(4)(X)$ of X to the Lorentz group $O(1, 3)$. Such a reduction is equivalent to finding a section of the associated fibre bundle with fibre $GL(4)/O(1, 3)$, denoted by $GL(4)/O(1, 3)(X)$. Steenrod showed that the latter bundle was fibre homotopy equivalent to the projective bundle $\mathbb{R}P^3(X)$ of all one-dimensional subspaces of the tangent bundle $\mathbb{R}^4(X) = T(X)$ of X. Therefore the Einstein bundle reduces to the Lorentz group iff there is a global section of $\mathbb{R}P^3(X)$ iff X admits a tangent line-bundle iff $T(X)$ splits off a line-bundle: $T(X) = \xi^3 \oplus \xi^1$. Markus showed that the obstruction to finding a tangent line bundle on X was essentially the Euler class e_X of X, a cohomology class in $H^4(X, \mathbb{Z})$. Therefore, if X is non-compact (implying that $H^4(X, \mathbb{Z}) = 0$), there is no obstruction. If X is compact and if $\langle X \rangle$ is the $\mathbb{Z}$-orientation class of X in $H_4(X, \mathbb{Z})$, $e_X\langle X \rangle = \chi(X)$, the Euler number of X and therefore the obstruction vanishes iff $\chi(X) = 0$.

By the above, to classify non-compact Lorentzian manifolds, one has first to classify non-compact four-manifolds up to diffeomorphism. Such a classification is impossible by a classic theorem of Markov (1958). Cobordism is a much weaker relation between manifolds. Two manifolds X_1 and X_2 of the same compactness type are called cobordant iff their disjoint sum is the boundary of a manifold of their compactness type. Any two non-compact manifolds are cobordant; for $X_1 = \partial(X_1 x [0, 1[)$ and $X_2 = \partial(X_2 x [0, 1[)$ and therefore if we remove the interior of coordinate disc in $X_1 x]0, 1[$ and in $X_2 x]0, 1[$ and identify the boundary spheres of the discs, we obtain a non-compact manifold with boundary $X_1 \cup X_2$. Cobordism is, however, non-trivial between compact manifolds and we shall classify closed, compact, orientable, smooth, four-manifolds of Euler number zero (which we call space-time manifolds) up to cobordisms defined below (which in dimension four include homotopy equivalence). The cobordism class of a space-time manifold will turn out to be uniquely determined by its topological signature sig( ): an integer invariant of the oriented manifold. The latter invariant has been discussed by Avez (1964) in the context of space-time geometry. Avez showed that static space-times and space-times of Petrov type III have trivial signature, providing an interpretation of sig( ) as an obstruction to geometric structure. As a by-product of our analysis, we obtain the following interpretations of signature. If X is any space-time manifold, there exists a compact hypermanifold H with $\partial H = X$ and H carries a one-frame field interior oriented on X. Equivalently, for any two space-time manifolds X_1 and X_2 there exists a compact hypermanifold H with boundary $\partial H = X_1 \cup X_2$ and H carries a one-frame field interior normal on X_1 and exterior normal on X_2. The signature plays the following role. An oriented space-time manifold X^+ is the *oriented* boundary $\partial_0 H^+ = X^+$ of a compact *oriented* hypermanifold H^+ iff $\mathrm{sig}(X^+) = 0$. Equivalently, two oriented space-time manifolds X_1^+ and X_2^+ form the oriented boundary of a compact oriented manifold H^+: $\partial_0 H^+ = X_1^+ \cup X_2^-$ iff $\mathrm{sig}(X_1^+) = \mathrm{sig}(X_2^+)$, H carrying a one-frame field interior normal on X_1 and exterior normal on X_2. Suppose that H is a hypermanifold for $X : \partial H = X$. Define a hypertrip (connecting points of X through hyperspace H) $\gamma : x_1 \rightsquigarrow x_2$ for $x_1, x_2 \in X$ as a path from x_0 to x_1 in H normal to X at x_1 and x_2 such that $\gamma^{-1}(X)$ consists of only the points x_1 and x_2. Because H admits a one-frame field, $T(H)$ splits off a trivial line-bundle: $T(H) = \xi^4 \oplus \xi^1$ where ξ^4 is a four-plane bundle tangent to X on ∂H. Recall (Husemoller, 1966) that a vector bundle ξ is orientable iff its first Stiefel–Whitney class $w_1(\xi)$ is trivial. Then, by the naturality properties of Stiefel–Whitney classes, $w_1(T(H)) = w_1(\xi^4) + w_1(\xi^1) = w_1(\xi^4)$. Thus H is orientable iff $T(H)$ is orientable iff ξ^4 is an orientable vector bundle. By our earlier remarks, X bounds a hypermanifold H with a transversely-oriented, oriented, four-plane bundle tangent to X on ∂H iff $\mathrm{sig}(X) = 0$. But such a ξ^4 is orientable iff its local orientations are preserved along all paths in H. Therefore if $\mathrm{sig}(X) \neq 0$, hypertrips may reverse the orientation of ξ^4, that is of X, since ξ^4 coincides with $T(X)$ on X. (In the hyperspace interpretation, taking a hypertrip between space-time points could reverse the hypertripper's sense of time or his sense of left and right relative to a trip through real space-time.) One can easily show that if a compact, oriented manifold has an orientation reversing autodiffeomorphism, it must have zero signature. We shall show that if X^+ is an oriented space-time manifold, there exists a compact, closed, oriented four-manifold Y^+ and a compact, oriented five-manifold H^+ with oriented boundary $\partial_0 H^+ = X^+ \cup 2Y^-$ (Fig. 1). If Y admits an orientation reversing autodiffeomorphism 'f' (implying that $\mathrm{sig}(Y^+) = \frac{1}{2}\,\mathrm{sig}(X^+) = 0$), we can join the two copies of Y along 'f' to

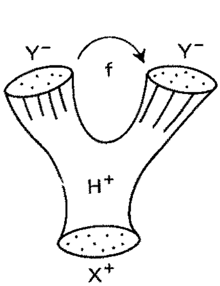

Figure 1.

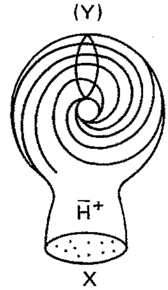

Figure 2.

obtain an oriented five-manifold $\bar{H}^+$ with oriented boundary X^+ (Fig. 2). The signature and the Euler number of an oriented four-manifold X are related by $\mathrm{sig}(X) \equiv \chi(X) \bmod (2)$ (a consequence of Poincaré duality on X). Thus the signature of a space-time manifold is always even. More complicated structures on X, such as spinor-structures, imply more detailed information about the signature. Briefly, a spinor-structure on a space and time-orientable space-time X is an extension of its Lorentz bundle $L_+\!\uparrow\!(X)$ to the group $\mathrm{Spin}_+(1, 3) = SL(2, \mathbb{C})$ (Porteous, 1969). Such an extension exists iff the canonical reduction of $L_+\!\uparrow\!(X)$ to $SO(3)$ (which always exists because $L_+\!\uparrow\!/SO(3)$ is contractible), extends to the group $\mathrm{Spin}(3) = SU(2)$. The former is a spin-structure on the real three-plane bundle ξ^3 associated to the principal bundle $SO(3)(X)$, where $T(X) = \xi^3 \oplus \xi^1$ splits as a direct consequence of the Lorentzian structure. ξ^3 admits a spin-structure iff $w_1(\xi^3)$ and $w_2(\xi^3)$ are trivial. But $w_1(\xi^3) = 0$ because X is space-orientable and $w_2(\xi^3) = w_2(T(X))$. Therefore a space-time manifold admits a spinor-structure iff it admits a spin-structure iff its Riemannian structural group $SO(4)$ extends to $\mathrm{Spin}(4)$. By a classic result of Rohlin (1951), the first Pontryagin number of a compact four-dimensional spin-manifold is divisible by 48. Therefore by a further result of Thom, the signature of a spinor-space-time manifold is divisible by 16. (Hence, for example, if we form the connected sum of two copies of $\mathbb{C}P^2$ to get a manifold of Euler number $+4$ and signature $+2$, and then add two disjoint one-handles ($S^3 \times I$) to kill the Euler number, we get a space-time manifold of signature $+2$ which cannot admit a spinor-structure.) We shall show that two spinor space-time manifolds form the spin-boundary of a compact five-dimensional spin-manifold iff they have the same signature.

Other topological properties of hypermanifolds of interest can be deduced from those of their bounding space-times. For example no space-time of nontrivial signature can bound a simply connected hypermanifold. If a space-time is an oriented boundary, it cannot bound a contractible hyperspace (contrast this to the case of the oriented boundary S^4 which bounds the closed five-disc. One can show that any four-dimensional oriented boundary bounding a contractible oriented manifold must have cohomology groups isomorphic to those of S^4).

Some Topological Invariants

In this section, X will denote a compact, closed, oriented, smooth $4k$-manifold with $k \geqslant 1$. Functorially associated with X are the graded homology modules $H_*(X, \mathbb{K})$ and the graded cohomology algebras $H^*(X, \mathbb{K})$ where $\mathbb{K} = \mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{R}$. The $\mathbb{K}$-cohomology and the $\mathbb{K}$-homology are paired to $\mathbb{K}$ by the Krönecker product. Because X is supposed orientable, $H_{4k}(X, \mathbb{K}) \cong \mathbb{K}$. We shall denote the generators of $H_{4k}(X, \mathbb{Z})$ and $H_{4k}(X, \mathbb{Z}_2)$ by $\langle X \rangle$ and $\langle X \rangle_2$ respectively. The following characteristic classes and numbers are also associated with X:

1. Stiefel-Whitney characteristic classes $w_i \in H^i(X, \mathbb{Z}_2)$ for $0 \leqslant i \leqslant 4k$ and Stiefel-Whitney numbers in $\mathbb{Z}_2$: $w_{i_1}^{r_1} \ldots w_{in}^{r_n} \langle X \rangle_2$ for $i_1 r_1 + \cdots + i_n r_n = 4k$. The top Stiefel-Whitney number $w_{4k} \langle X \rangle_2$ is related to the Euler number $\chi(X)$ by $w_{4k} \langle X \rangle_2 = \chi(X) \bmod (2)$ (the residue class of $\chi(X) \bmod (2)$).
2. Pontryagin characteristic classes $P_i \in H^{4i}(X, \mathbb{Z})$ for $0 \leqslant i \leqslant k$ and Pontryagin numbers (integers) $P_{i_1}^{r_1} \ldots P_{in}^{r_n}$ for $i_1 r_1 + \cdots + i_n r_n = k$. The Pontryagin classes reduced mod(2) are related to the Stiefel-Whitney classes by $[P_i] \bmod (2) = w_{2i}^2$, and therefore for the case $k = 1$ we get $w_2{}^2 \langle X \rangle_2 = (P_1 \langle X \rangle) \bmod (2)$.
3. The Hirzebruch signature (Hirzebruch, 1966) of X, sig (X) (sometimes called the index), which is defined as follows: The cohomology product $H^{2k}(X, \mathbb{R}) \otimes H^{2k}(X, \mathbb{R}) \to H^{4k}(X, \mathbb{R})$ induces a symmetric, non-degenerate, bilinear form on the vector space $H^{2k}(X, \mathbb{R})$ defined by $b_X(x,y) = x \,.\, y \langle X \rangle$ for $x, y \in H^{2k}(X, \mathbb{R})$ where $\langle X \rangle$ is the basis of $H^{2k}(X, \mathbb{R})$. $\mathrm{sig}(X)$ is defined as the signature of X, that is, $\mathrm{sig}(X) = p - n$ where p is the number of positive eigenvalues and n the number of negative eigenvalues of b_X. Because $p + n = \beta_{2k}(X)$ (where $\beta_i(X) = \dim_{\mathbb{R}} (H^i(X, \mathbb{R}))$ is the ith Betti number of X), $\mathrm{sig}(X) \equiv \beta_{2k}(X) \bmod (2)$. But Poincaré duality, $H^{4k-q}(X, \mathbb{R}) \cong H_q(X, \mathbb{R})$, and $H^q(X, \mathbb{R}) \cong H_q(X, \mathbb{R})^*$ implies that $\chi(X) = \Sigma_{i=0}^{4k} (-1)^i \beta_i(X) \equiv \beta_{2k}(X) \bmod (2)$. Therefore $\mathrm{sig}(X) \equiv \chi(X) \bmod (2)$. Note that if X admits an orientation reversing autodiffeomorphism 'f', $\mathrm{sig}(X) = 0$. For then, if f^* denotes the cohomology automorphism induced by f and f_* denotes the homology automorphism, $b_X(f^*x, f^*y) = f^*x \,.\, f^*y \langle X \rangle = x \,.\, y f_* \langle X \rangle = x \,.\, y(-\langle X \rangle) = -b_X(x, y)$. $(f_* \langle X \rangle = -\langle X \rangle$ because f reverses the orientation of $X)$. Therefore $b_X(f^*x, f^*y) = -b_X(x, y)$ implying that f^* exchanges the negative definite and the positive definite subspaces of $H^{2k}(X, \mathbb{R})$ and therefore $p = n$ or $\mathrm{sig}(b_X) = 0$.
4. The Hirzebruch L-genus and the Hirzebruch A-genus (Hirzebruch, 1966) of X. These are characteristic polynomials in the Pontryagin classes of X and split into components $L_i \in H^{4i}(X, \mathbb{Z})$ and $A_i \in H^{4i}(X, \mathbb{Z})$. It is a celebrated theorem of Hirzebruch that the L_k-genus evaluated on the class $\langle X \rangle$ is the signature of X: $L_k \langle X \rangle = \mathrm{sig}(X)$. For the special case $k = 1$, this is the theorem of Rohlin and Thom: $L_1 \langle X \rangle = \frac{1}{3} \,.\, P_1 \langle X \rangle = \mathrm{sig}(X)$. For a spin-manifold ($w_1 = 0$ and $w_2 = 0$) the characteristic number $A_k \langle X \rangle$ is an integer which is even if k is odd. Therefore in the special case $k = 1$, $A_1 \langle X \rangle = \frac{1}{24} P_1 \langle X \rangle$ is an even integer, that is, $P_1 \langle X \rangle \equiv 0 \bmod (48)$. Then by the theorem of Rohlin Thom, $\mathrm{sig}\,(X) \equiv \bmod (16)$.

Cobordism Relations

We shall need the following cobordism relations between closed, compact manifolds (Strong, 1969).

Definition 1. Two manifolds X_1 and X_2 are called cobordant in the unoriented sense iff there exists a compact manifold H with $\partial H = X_1 \cup X_2$. The above cobordism relation will be abbreviated to O-cobordism. The sum and the product of manifolds induce the structure of a $\mathbb{Z}_2$-algebra on the set M_O of O-cobordism classes. The algebra is graded into subgroups of O-cobordism classes of n-manifolds $M_O{}^n$. Its structure was determined by Thom (1954) who showed that two manifolds are O-cobordant iff they have the same Stiefel-Whitney numbers. Because Stiefel-Whitney classes are topological invariants, homotopically equivalent manifolds are O-cobordant. There is the following cobordism relation between oriented manifolds.

Definition 2. Two oriented manifolds X_1^+ and X_2^+ are called cobordant in the oriented sense iff there exists a compact oriented manifold H^+ with oriented boundary $\partial_0 H^+ = X_1^+ \cup X_2^-$ (SO-cobordism).

The set of SO-cobordism classes of oriented manifolds M_{SO} is a ring graded by the subgroups of SO-cobordism classes of n-manifolds M_{SO}^n. The structure of M_{SO} was first fully determined by Wall (1960). A necessary and sufficient condition for two oriented manifolds to be SO-cobordant is that they have the same Stiefel-Whitney and Pontryagin numbers. Note that because the L_k genus is a function of Pontryagin classes, the signature is an SO-cobordism invariant. In a more restrictive version of cobordism, the Euler number is a cobordism invariant. This is vector cobordism defined by Reinhart (1963).

Definition 3. Two manifolds X_1 and X_2 are called vector cobordant in the unoriented sense iff there exists a manifold H with $\partial H = X_1 \cup X_2$ and H carries a one-frame field interior oriented on X_1 and exterior oriented on X_2 (OV-cobordism).

The set M_{OV} of OV-cobordism classes of manifolds is a graded group whose structure was determined by Reinhart who showed that two manifolds are OV-cobordant iff they have the same Stiefel-Whitney and Euler numbers. The oriented version of vector cobordism is SV-cobordism also due to Reinhart.

Definition 4. Two oriented manifolds X_1^+ and X_2^+ are called vector cobordant in the oriented sense iff there exists an oriented manifold H^+ with oriented boundary $\partial_0 H^+ = X_1^+ \cup X_2^-$ and H carries a one-frame-field interior normal on X_1 and exterior normal on X_2.

Reinhart determined the structure of the SV-cobordism group M_{SV} and showed that two oriented manifolds are SV-cobordant iff they have the same Stiefel-Whitney, Euler and Pontryagin numbers. There is also a cobordism relation between compact spin-manifolds.

Definition 5. Two spin-manifolds X_1^{+,s_1} and X_2^{+,s_2} are called spin-cobordant iff there exists a compact spin-manifold $H^{+,S}$ with spin-boundary $\partial_s H^{+,S} = X_1^{+,s_1} \cup X_2^{-,\bar{s}_2}$ where $\bar{s}$ denotes the spin-structure opposite to s.

The set M_{Spin} of spin-cobordism classes of compact, closed spin-manifolds is a graded ring whose structure was determined by Anderson, Brown & Peterson (1967). The function $|X|_{\text{Spin}} \to |X|_{SO}$ is a ring homomorphism $M^n_{\text{Spin}} \to M^n_{SO}$ whose kernel is non-trivial only in dimensions $\equiv 1$ or $2 \bmod (8)$.

The Cobordism of Space-Time Manifolds

Because the Euler number is a vector cobordism invariant and space-time manifolds are defined to be those compact, closed, orientable four-manifolds with Euler number zero, vector cobordism is a natural tool to use in their classification. Any four-manifold vector cobordant to a space-time manifold is itself a space-time manifold. We shall also examine the other cobordism classes of space-time manifolds although these cobordism classes contain four-manifolds which are not space-times. For example the space-time manifold $S^3 \times S^1$ is SO-cobordant to S^4 which is not a space-time. This is also a Spin-cobordism ($w_2(S^3 \times S^1) = w_2(S^3) = 0$ since by a theorem of Wu if Z is a compact manifold of dimension $n \equiv 3 \bmod (4)$, $w_{n-1}(Z) = 0$; also $w_2(S^4) = 0$ because S^4 imbeds in $\mathbb{R}^5$ implying that the total Stiefel-Whitney class $W(S^4) = \Sigma_i\, w_i(S^4)$ satisfies $W(S^4).\ 1 = W(\mathbb{R}^5) = 1$ and therefore $w_i(S^4) = 0$ for $i \geqslant 1$.) We shall construct groups of cobordism classes of space-time manifolds and examine their structure.

Proposition 1. Suppose that $\chi : M^4_{SV} \to \mathbb{Z}$ is the mapping $\chi : |X|_{SV} \to \chi(X)$ the Euler number of X; χ is a group homomorphism.

Proof. χ is a function because if $|X|_{SV} = |Y|_{SV}$, $\chi(X) = \chi(Y)$; χ is a group homomorphism because one can show that $\chi(X \cup Y) = \chi(X) + \chi(Y)$ by the fact that $H_*(X \cup Y; \mathbb{R}) = H_*(X; \mathbb{R}) \oplus H_*(Y; \mathbb{R})$.

By Proposition 1, $\ker(\chi)$ is a subgroup of M^4_{SV} and it consists precisely of the SV-cobordism classes of oriented space-time manifolds. We shall denote this group by $K(SV)$. Consider the following commutative diagram:

$$\begin{array}{ccc}
M^4_{SV} & \xrightarrow{f_s^{\,v}} & M^4_{SO} \\
\Big\downarrow{\scriptstyle f_v^{\,0}} & & \Big\downarrow{\scriptstyle f^0} \\
M^4_{OV} & \xrightarrow[f_0^{\,v}]{} & M^4_{SO}
\end{array}$$

The homomorphisms $f_s^{\,v}$ and $f_0^{\,v}$ 'forget' the vector structure, $f_v^{\,0}$ forgets the orientation in vector cobordism and f^0 forgets the orientation in ordinary cobordism. In detail, $f_s^{\,v} : |X|_{SV} \to |X|_{SO}$; $f_0^{\,v} : |X|_{OV} \to |X|_O$; $f_v^{\,0}$: $|X|_{SV} \to |X|_{OV}$ and $f^0 : |X|_{SO} \to |X|_O$. The groups $K(OV)$, $K(SO)$ and $K(O)$

are defined as $f_v^0(K(SV))$, $f_s^v(K(SV))$ and $f^0(K(SO))$ respectively. It turns out that $K(OV)$ and $K(O)$ are trivial groups.

Proposition 2. $K(SV)$ is a subgroup of $\ker(f_v^0)$.

Proof. We shall show that if X is any closed, compact, orientable, four-manifold of even Euler number, then X has trivial Stiefel-Whitney numbers. The Stiefel-Whitney numbers of X are just:

$$w_1^4\langle X\rangle_2, \quad w_1^2 . w_2\langle X\rangle_2, \quad w_1 . w_3\langle X\rangle_2, \quad w_2^2\langle X\rangle_2 \quad \text{and} \quad w_4\langle X\rangle_2$$

But recall that X is orientable iff $w_1 = 0$; consequently we need only examine the numbers $w_4\langle X\rangle_2$ and $w_2^2\langle X\rangle_2$. By hypothesis, $w_4\langle X\rangle_2 = \chi(X) \bmod (2) = 0$. Now $w_2^2\langle X\rangle_2 = P_1\langle X\rangle \bmod (2)$. The theorem of Rohlin-Thom: $P_1\langle X\rangle = 3\,\mathrm{sig}(X)$ gives $P_1\langle X\rangle \bmod (2) = \mathrm{sig}(X) \bmod (2)$ and Poincaré duality gives $\mathrm{sig}(X) \equiv \chi(X) \bmod (2)$. Therefore $w_2^2\langle X\rangle_2 = \chi(X) \bmod (2) = 0$. Hence any four-manifold of even Euler number is O-cobordant and in particular $K(SV)$ must be a subgroup of $\ker(f_v^0)$.

Corollary 3. Any space-time manifold is the boundary of a compact five-manifold which carries a one-frame field interior normal on the space-time. Equivalently, if X_1 and X_2 are space-time manifolds, there exists a compact five-manifold H with boundary $\partial H = X_1 \cup X_2$ and H carries a one-frame field interior normal on X_1 and exterior normal on X_2. The next proposition deals with the structure of the group $K(SV)$.

Proposition 4. $K(SV)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$.

Proof. Suppose that n is even and consider the complex projective plane $\mathbb{C}P^n = S^{2n+1}/S^1$. The cohomology algebra $H^*(\mathbb{C}P^n; R)$ is the truncated polynomial algebra of length $2n$ on a two-dimensional generator. Hence the cohomology product: $H^n(\mathbb{C}P^n;\mathbb{R}) \otimes H^n(\mathbb{C}P^n;\mathbb{R}) \to H^{2n}(\mathbb{C}P^n;\mathbb{R})$ induces the bilinear form $b_{\mathbb{C}P^n}(x.g, y.g) = xy$ for g the generator g_0^k of $H^n(\mathbb{C}P^n,\mathbb{R})$, $x, y \in \mathbb{R}$, g_0 the two-dimensional generator of $H^*(\mathbb{C}P^n;\mathbb{R})$ and $n = 2k$. Therefore $\mathrm{sig}(\mathbb{C}P^{2k}) = +1 . \chi(\mathbb{C}P^{2k}) = 2k+1$, therefore, for $k = 1$ we get $\mathrm{sig}(\mathbb{C}P^2) = +1$ and $\chi(\mathbb{C}P^2) = +3$. The disjoint sum $\mathbb{C}P^2 \cup \mathbb{C}P^2$ has signature $+2$ and Euler number $+6$. Connect the manifold by excising a four-disc from each copy of $\mathbb{C}P^2$ and identifying one of the resulting boundary three-spheres with $S^3 \times 0$ and the other with $S^3 \times 1$ in the cylinder $S^3 \times I$. The resulting manifold has Euler number $+4$. By adding a pair of disjoint one-handles, one $S^3 \times I$ to the remaining part of each $\mathbb{C}P^2$, we obtain a compact, closed, orientable four-manifold X_0 of Euler number zero and signature $+2$. (The fact that X_0 has Euler number zero is due to our construction. Because X_0 was obtained from $2 . \mathbb{C}P^2$ by spherical modification and spherical modification preserves oriented cobordism class, X_0 has the same signature as $2 . \mathbb{C}P^2$. It can be shown that $\mathrm{sig}(X \cup Y) = \mathrm{sig}(X) + \mathrm{sig}(Y)$.) Recall that if X is a space-time manifold it has even signature: $\mathrm{sig}(X) \equiv \chi(X) \bmod (2) = 0$. We shall show that $|X|_{SV} = \frac{1}{2}\,\mathrm{sig}(X) . |X_0|_{SV}$. Now X, X_0 and therefore $\frac{1}{2}\,\mathrm{sig}(X) . X_0$ have Euler number zero and therefore trivial Stiefel–Whitney numbers. Moreover, $\mathrm{sig}(\frac{1}{2}\,\mathrm{sig}(X).X_0) = \frac{1}{2}\,\mathrm{sig}(X).\mathrm{sig}(X_0) = \mathrm{sig}(X)$. Therefore, by the theorem of Rohlin–Thom, $P_1\langle X\rangle = P_1\langle \frac{1}{2}\,\mathrm{sig}(X).X_0\rangle$. The assertion now follows because X and $\frac{1}{2}\,\mathrm{sig}(X).X_0$ have the same Stiefel-Whitney, Euler and Pontryagin numbers. It is clear that $|X_0|_{SV}$ generates $K(SV)$ and that the function $|X|_{SV} \to \frac{1}{2}\,\mathrm{sig}(X)$ is a group isomorphism $K(SV) \cong \mathbb{Z}$.

Corollary 5. $K(SO) = 2\,.\,M^4_{SO}$.

Proof. We shall show in the next proposition that $|\mathbb{C}P^2|_{SO}$ generates M^4_{SO}. The corollary will then be a direct consequence of the fact that $|X_0|_{SO} = 2\,|\mathbb{C}P^2|_{SO}$ (because X_0 is obtained from $2\,.\mathbb{C}P^2$ by spherical modification). Another way to prove the corollary is as follows. Consider the mappings ${}_2\chi^n : M^n_{SO} \to \mathbb{Z}_2$ defined by ${}_2\chi^n : |X|_{SO} \to \chi(X) \bmod (2)$. If $n \not\equiv 3 \bmod (4)$, ${}_2\chi^n$ is a function. For if $|X|_{SO} = |Y|_{SO}$, there is an oriented, compact, $(n+1)$-manifold H^+ with oriented boundary $\partial_0 H^+ = X^+ \cup Y^-$. Glue H^+ and H^- by identifying the manifolds X and Y in their boundary components to obtain a closed, compact, oriented manifold $H\&H$ with $\chi(H\&H) = 2\chi(H) - \chi(X) - \chi(Y)$. Because $(n+1) \not\equiv 0 \bmod (4)$, $\chi(H\&H)$ is even, implying that $\chi(X) \equiv \chi(Y) \bmod (2)$. In dimensions $n \equiv 0 \bmod (4)$, the functions ${}_2\chi^n$ are non-trivial. They are group homomorphisms because $\chi(X \cup Y) = \chi(X) + \chi(Y)$. In the special case of dimension four, we claim that $\ker({}_2\chi^4) = K(SO) = 2M^4_{SO}$. Obviously $K(SO)$ is a subgroup of $\ker({}_2\chi^4)$, conversely, if X has even Euler number, one can modify X into a manifold of Euler number zero. (If $\chi(X) = 2k > 0$, add k disjoint one-handles. If $\chi(X) = 2k < 0$, excise the interior of a region diffeomorphism to $D^3 \times S^1$ and identify the resulting boundary $S^2 \times S^1$ to the boundary $S^2 \times S^1$ of $S^2 \times D^2$. This spherical modification increases the Euler number of X by $+2$ and is repeated k-times (taking disjoint regions in turn).) The modified manifold is in the SO-cobordism class of X. Therefore $K(SO) = \ker({}_2\chi^4)$. Any class in $2\,.\,M^4_{SO}$ is obviously of even Euler number. Conversely, if a manifold has even Euler number, we saw above that it must be O-cobordant and hence lies in $\ker(f^0 : M^4_{SO} \to M_O{}^4)$. By a theorem of Wall, $\ker(f^0) = 2\,.\,M^4_{SO}$. (A similar result holds in each dimension.)

Proposition 6. The function $\mathbf{sig} : M^4_{SO} \to \mathbb{Z}$, $\mathbf{sig} : |X|_{SO} \to \mathrm{sig}(X)$ is a group isomorphism.

Proof. Because signature is an oriented cobordism invariant, **sig** is a function. It is a group homomorphism because it is additive over disjoint unions. Ker(**sig**) = 0, because if an oriented, closed, compact four-manifold X has zero signature, $\mathrm{sig}(X) = 0$, then $P_1\langle X\rangle = 0$ and the fact that $\mathrm{sig}(X) = \chi(X) \bmod (2)$ implies that $\chi(X)$ is even and hence that X has trivial Stiefel-Whitney numbers. Therefore $|X|_{SO} = 0$. **sig** is an epimorphism because for any integer 'm' there exists a closed, compact, oriented manifold of signature m. (If $m = 0$, take $X = S^4$; if $m > 0$, take $X = m\,.\,\mathbb{C}P^2$ and if $m < 0$, take $X = |m|\,.\,\mathbb{C}P^{2(-)}$.) Because $\mathbf{sig}^{-1}(1) = |\mathbb{C}P^2|_{SO}$, $|\mathbb{C}P^2|_{SO}$ freely generates M^4_{SO}.

Recall that in Proposition 2 we proved that $K(SV)$ is a subgroup of $\ker(f_v{}^0)$. The next proposition deals with the structure of $\ker(f_v{}^0)$ using results of Wall and Reinhart.

Proposition 7. If n is even and if $|X|_{SV} \in \ker(f_v{}^0 : M^n_{SV} \to M^n_{OV})$, there exists an oriented, closed, compact n-manifold Y^+ such that $|X|_{SV} = 2|Y|_{SV} + m \, . \, |S^4|_{SV}$, where m is given by $\frac{1}{2}(\chi(X) - 2\chi(Y))$.

Proof. If $|X|_{SV} \in \ker(f_v{}^0)$, $f_s{}^v(|X|_{SV}) = |X|_{SO} \in \ker(f^0)$. For $f^0 \circ f_s{}^v(|X|_{SV}) = f_0{}^v \circ f_v{}^0(|X|_{SV}) = 0$. By the theorem of Wall, $\ker(f^0) = \mathrm{Im}(2_*)$ where 2_* denotes multiplication by two in the abelian group M^n_{SO}. Thus there exists a closed, compact, oriented n-manifold Y such that $|X|_{SO} = 2|Y|_{SO}$ implying that $|X|_{SV} - 2|Y|_{SV} \in \ker(f_s{}^v)$. By a result of Reinhart, $\ker(f_s{}^v)$ is free cyclic on $|S^n|_{SV}$ for even n. Therefore there exists a integer m such that $|X|_{SV} = 2|Y|_{SV} + m \, . \, |S^n|_{SV}$. Because the Euler number is a vector cobordism invariant m is given by $\chi(X) = 2\chi(Y) + 2m$. ($\chi(S^n) = 2$.)

Corollary 8. If $|X|_{SV} \in K(SV)$, $|X|_{SV} = 2|Y|_{SV} - 2\chi(Y)|S^4|_{SV}$ for some closed, compact, oriented four-manifold Y unique up to SO-cobordism.

Proof. The result follows directly from the proposition. Y is unique up to SO-cobordism because if $2|Y'|_{SV} - \chi(Y')|S^4|_{SV} = |X|_{SV} = 2|Y|_{SV} - \chi(Y)|S^4|_{SV}$ then $2 \, . \, |Y'|_{SV} - 2|Y|_{SV} = (\chi(Y') - \chi(Y))|S^4|_{SV}$ and therefore $|Y|_{SO} = |Y'|_{SO}$ because $|S^4|_{SO} = 0$ and M^4_{SO} has no two-torsion ($\ker(2_*) = 0$).

Note that for a given space-time manifold X, $|X|_{SO} = \mathrm{sig}(X)|\mathbb{C}P^2|_{SO}$ so that a possible choice for Y is $Y_0 = \frac{1}{2}\,\mathrm{sig}(X) . \mathbb{C}P^2$. Therefore by Corollary 8 one can write $|X|_{SV} = \mathrm{sig}(X) . |\mathbb{C}P^2|_{SV} - \frac{3}{2}\,\mathrm{sig}(X) . |S^4|_{SV}$. We next consider the spin cobordism of spinor space-time manifolds.

Proposition 9. Two spinor space-time manifolds are spin-cobordant iff they are SO-corbordant iff they have the same signature. Any four-dimensional spin-manifold is SO-cobordant to a space-time manifold.

Proof. (i) We noted earlier that the kernel of the canonical homomorphism $M^n_{\mathrm{Spin}} \to M^n_{SO}$ was zero in dimensions not congruent to 1 or 2 mod(8). Therefore $M^4_{\mathrm{Spin}} \to M^4_{SO}$ is a monomorphism and two spinor space-times are spin-cobordant iff they are SO-cobordant. (ii) A spin-manifold is defined to have $w_1 = 0$ and $w_2 = 0$. Therefore, because $\chi(X) \bmod(2) = w_2{}^2\langle X\rangle_2$ for a four-manifold, a four-dimensional spin-manifold must have even Euler number. Hence one can modify X into a space-time manifold in the oriented cobordism class of X.

It is clear that the topology of a space-time manifold must place restrictions on the topology of any manifold that it bounds. There are the following immediate results.

Proposition 10. (i) If X is a space-time manifold with non-zero signature, X is not the boundary of any simply-connected hypermanifold. (ii) If $\mathrm{sig}(X) = 0$ and therefore X bounds an oriented hypermanifold, H can be simply-connected but cannot be contractible.

Proof. (i) If X bounds a simply-connected manifold H, H must also be orientable which implies that X is SO-cobordant and therefore that $\mathrm{sig}(X) = 0$. (ii) If X is the oriented boundary of a simply-connected manifold H, H cannot be contractible. Because $\chi(X) = 0 = 2(1 - \beta_1(X)) + \beta_2(X)$ and $\beta_2(X) \geqslant 0$,

$\beta_1(X) \neq 0$ and therefore $H^1(X, \mathbb{R}) \neq 0$. Therefore by looking at the exact cohomology sequence of the pair (H, X):

$$\ldots \to H^1(H, X; \mathbb{R}) \xrightarrow{j^*} H^1(H; \mathbb{R}) \xrightarrow{i^*} H^1(X; \mathbb{R}) \xrightarrow{d^*} H^2(H, X; \mathbb{R}) \to \ldots$$

and using the fact that H simply-connected implies that $H^1(H; \mathbb{R}) = 0$, the connecting homomorphism d^* is a monomorphism. Thus $H^1(X, \mathbb{R}) \neq 0$ implies that $H^2(H, X; \mathbb{R}) \neq 0$. The Lefchetz duality theorem ($H^q(Z, \partial Z; \mathbb{K}) \cong H_{n-q}(Z; \mathbb{K})$ for an n-manifold Z orientable over $\mathbb{K}$) implies that $H_3(H; \mathbb{R}) \neq 0$ and hence that H cannot be contractible. Note that the space-time $X = S^3 \times S^1$ is an oriented boundary, $X = \partial_0(D^2 \times S^3)$ and that $D^2 \times S^3$ is simply connected. This shows that a compact, oriented space-time *can* bound a simply connected hypermanifold.

It would be interesting to define a cobordism theory using pseudo-Riemannian structural groups instead of the Riemannian structural groups $O(n)$ (unoriented cobordism); $SO(n)$ (oriented cobordism) and $\mathrm{Spin}(n)$ (Spin cobordism). For example the following cobordism relation gives a cobordism between psuedo-Riemannian *structures* on manifolds. Recall that a time-oriented Lorentzian structure on a manifold is equivalent to putting a one-frame field on the manifold, if the manifold is compact the vector field defines an action of the group $\mathbb{R}$ on X. Consider the following relation.

Definition 6. Suppose that (X_1, a_1) and (X_2, a_2) are compact manifolds with $\mathbb{R}$-actions a_1 and a_2. Then (X_1, a_1) and (X_2, a_2) are called cobordant iff there exists a compact manifold (H, A) with boundary $\partial H = X_1 \cup X_2$ such that the $\mathbb{R}$-action A restricts to a_1 on X_1 and to a_2 on X_2.

Such cobordism relations were introduced by Conner and Floyd (1964). who were interested in the actions of the groups $\mathbb{Z}_p$ on manifolds. If one restricts attention to a given manifold X, one obtains the cobordism group $A_{\mathbb{R}}(X)$ of $\mathbb{R}$-actions on X. It would be interesting to calculate such a group for a space-time manifold X: A cobordism of two $\mathbb{R}$-actions being viewed as a deformation of the corresponding Lorentzian structures into each other.

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